Point Form Electrodynamics and the Construction of Conserved Current Operators

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Abstract

A general procedure for constructing conserved electromagnetic current operators, for both finite and infinite degree of freedom systems, is given. A four-momentum operator consisting of matter, photon, and electromagnetic interactions is assumed to be polynomial in photon creation and annihilation operators. Commutators of the four-momentum operator with the four-vector potential operator at the space-time point zero give the electromagnetic field tensor and current operator at the space-time point zero. In this construction there are no equations of motion to be satisfied and field operators at an arbitrary space-time point-defined as the four-momentum translates of the corresponding operators at the space-time point zero-are not local operators. Several examples are given to show how the construction is carried out[1].

1 Introduction

In quantum field theory, if the Lagrangian for a system is invariant under an internal symmetry, a current is generated, which however, is not conserved

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unless the equations of motion are satisfied. The goal of this paper is to construct electromagnetic current operators for strongly interacting matter in which there are no equations of motion to be satisfied, and nevertheless the currents are conserved. The context for this work is point form relativistic quantum mechanics, in which all of the interactions of the system are in the four-momentum operator, and the Lorentz generators are kinematic, that is, contain no interactions. The four-momentum operator is the sum of three terms, the four-momentum operator for matter, for photons, and for the electromagnetic coupling. In the previous paper[1] only the four-momentum operator for photons was used in analyzing the relationship of the photon creation and annihilation operators to the free four-vector potential operator.

Since the four-momentum operator for matter commutes with the photon four-momentum operator, the four-vector potential field, defined as the space-time translation of the operator from the space-time point zero to an arbitrary space-time point x, will continue to be local if the four-momentum operator for the electromagnetic coupling is zero. But if the four-momentum operator for the electromagnetic coupling is not zero, fields in general, defined as space-time translates of operators from the space-time point zero will not be local. As will be seen in section 3 this causes no problems, since the primary objects of interest are the fields at the space-time point zero. The electromagnetic field tensor at the space-time point zero will be shown to be the commutator of the four-vector potential at the space-time point zero with the four-momentum operator and the current operator at the space-time point zero the commutator of the field tensor with the four-momentum operator. And the check that the current is conserved will again involve the computation of the commutator of the current operator with the four-momentum operator, which in this case should give zero. These quantities can all be defined at an arbitrary space-time point, but they are then derived rather than fundamental quantities. In a sense it is a (relativistic) Schrödinger picture, rather than a Heisenberg picture, as is usually the case in quantum field theory that is being used.

While quantum field theory will be used as a guide in computing current operators, the real motivation for constructing conserved current operators arises from relativistic quantum mechanics for systems with a few degrees of freedom, in which, in the point form, a four-momentum operator for matter is constructed from a mass operator. Calculating observables such as form factors involves integrating eigenfunctions of the mass operator with kernels of current operators, which are usually not conserved. In the point form the third component of the current operator acting on mass operator eigenstates in the Breit frame involves current conservation, which is not satisfied unless auxiliary currents (whose form need not be known) are introduced[2]. The goal of this paper is to compute current operators that are conserved, and for which it is not necessary to introduce other currents to satisfy current conservation.

In section 2 the main features of point form relativistic quantum mechanics are reviewed, while in section 3 the ideas of point form electrodynamics are laid out. In particular it is shown that matrix elements of the current and electromagnetic field operators recover the classical Maxwell equations. Then in section 4 examples from quantum field theory as well as for systems with a finite number of degrees of freedom are given.

2 Point Form Relativistic Quantum Mechanics

In point form relativistic quantum mechanics all interaction are put in the four-momentum operator. The Lorentz generators contain no interactions so that the point form is manifestly Lorentz covariant. In fact the Lorentz generators will almost never be used; instead global Lorentz transformations will be used to define the transformation properties of operators and states. In particular the Lorentz transformation properties of photons, worked out in the previous paper[1], will be used in analyzing the electromagnetic interactions. If U_Λ is a unitary operator representing the Lorentz transformation Λ, then the four-momentum operator must satisfy the following point form equations:

$$\begin{aligned} [P_\mu, P_\nu] &= 0 &(1)\\ U_\Lambda P_\mu U_\Lambda^{-1} &= (\Lambda^{-1})^\nu_\mu P_\nu. &(2)\end{aligned}$$

These equations are simply one way of writing the Poincaré commutation relations in which the relations of the four-momentum operators are emphasized. The mass operator is defined to be $M := \sqrt{P \cdot P}$ and must have a nonnegative spectrum.

Since the four-momentum operators are the generators of space-time translations, they can be used to define a relativistic Schrödinger equation,

$$i\frac{\partial \Psi_x}{\partial x^\mu} = P_\mu \Psi_x \qquad (3)$$

where Ψ_x is an element of the Hilbert space and $x(= x_\mu)$ is a space-time point. From Eq.3 it follows that Ψ_x satisfies a generalized Klein-Gordan equation,

$$(\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} + M^2)\Psi_x \quad = \quad 0, \tag{4}$$

where M is the mass operator. It is Eqs. 3,4 that take the place of equations of motion. If P_μ has no explicit space-time dependence, then Eq.3 can be written as an eigenvalue equation for the four-momentum operator:

$$P_\mu \Psi \quad = \quad p_\mu \Psi. \tag{5}$$

Assume now that P_μ is the sum of matter, photon, and electromagnetic four-momentum operators, and satisfies Eqs.1,2. The matter four-momentum operator, $P_\mu(matter)$, is assumed to be independent of photon creation and annihilation operators, while the photon four-momentum operator, $P_\mu(\gamma)$, is proportional to $a^\dagger a$ (see Eq.25, reference [1]). The main point of this paper is to show how to calculate a conserved current operator if the electromagnetic four-momentum operator, $P_\mu(em)$, is a polynomial in photon creation and annhilation operators, even if the matter four-momentum operator contains (strong) interactions.

3 Point Form Electrodynamics

The fundamental quantities in point form electrodynamics are the four-momentum operator, P_μ, and the four-vector potential operator at the space-time point zero, $A_\mu(0)$ (which itself is defined via the photon creation and annihilation operators, see Eq.28, reference [1]). From these quantities the electromagnetic field tensor at the space-time point zero is defined as

$$iF_{\mu\nu}(0) := [A_\mu(0), P_\nu] - [A_\nu(0), P_\mu], \tag{6}$$

and is antisymmetric in μ and ν. From this definition it follows that the electromagnetic field tensor at the space-time point x satisfies the usual relationship with the four-vector potential:

$$\begin{aligned} iF_{\mu\nu}(x): \quad &= \quad ie^{iP\cdot x}F_{\mu\nu}(0)e^{-iP\cdot x} && (7)\\ &= \quad [A_\mu(x), P_\nu] - [A_\nu(x), P_\mu] && (8)\\ &= \quad i(\frac{\partial A_\mu(x)}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu}), && (9) \end{aligned}$$

where $A_\mu(x) = e^{iP\cdot x}A_\mu(0)e^{-iP\cdot x}$. It should be noted that if the electromagnetic four-momentum operator is nonzero, then both the four-vector potential and the electromagnetic field tensor will not in general be local. Also the correct Lorentz transformation properties of the field quantities all follow from the Lorentz properties of operators at the space-time point zero.

The electromagnetic current operator at the space-time point zero can then be defined as

$$iJ_\mu(0) := [F_{\mu\nu}(0), P^\nu], \tag{10}$$

from which it follows that

$$\begin{aligned} iJ_\mu(x) &= ie^{iP\cdot x}J_\mu(0)e^{-iP\cdot x} \quad (11) \\ &= e^{iP\cdot x}[F_{\mu\nu}(0), P^\nu]e^{-iP\cdot x} \\ &= i\frac{\partial F_{\mu\nu}(x)}{\partial x_\nu}. \quad (12) \end{aligned}$$

Such a current is always conserved. This follows directly from the antisymmetry of $F_{\mu\nu}(x)$ and Eq.12, but it is worthwhile deriving current conservation as a commutator relation, namely $[P^\mu, J_\mu(0)] = 0$, since this form of current conservation will be used in the following section.

Now, $\partial J_\mu(x)/\partial x_\mu = 0$ is equivalent to $[P^\mu, J_\mu(0)] = 0$. But

$$\begin{aligned} i[P^\mu, J_\mu(0)] &= [P^\mu, [F_{\mu\nu}(0), P^\nu]] \\ &= -[F_{\mu\nu}(0), [P^\nu, P^\mu]] - [P^\nu, [P^\mu, F_{\mu\nu}(0)]] \\ &= [P^\nu, [F_{\mu\nu}(0), P^\mu]] \\ &= -[P^\mu, [F_{\mu\nu}(0), P^\nu]] \\ &= 0. \end{aligned} \tag{13}$$

To arrive at this result the Jacobi identity for three operators was used, as well as the antisymmetry of $F_{\mu\nu}(0)$ and the point form Eq.1. Thus, any current operator defined by Eq.10 is conserved if the electromagnetic field tensor is antisymmetric and the four-momentum operators commute with one another.

Because the fields are not local, gauge transformations are defined by adding an operator (rather than c-number) to $A_\mu(x)$:

$$A'_\mu(0) := A_\mu(0) + i[P_\mu, \chi(0)], \tag{14}$$

where $\chi(0)$ is a Lorentz scalar operator. If $\chi(0)$ is space-time translated to $\chi(x)$, it follows that

$$
\begin{aligned}
A'_\mu(x) &= A_\mu(x) + \frac{\partial\chi(x)}{\partial x^\mu}, && (15)\\
iF'_{\mu\nu}(0): &= [A'_\mu(0), P_\nu] - [A'_\nu(0), P_\mu] && (16)\\
&= iF_{\mu\nu}(0),
\end{aligned}
$$

so that the electromagnetic field tensor is gauge invariant. Since the gauge transformation defined in Eq.14 involves a scalar operator rather than a c-number function, as is the case for local fields, the link to gauge transformation in terms of photon creation and annihilation operators is different than that given in reference [1]. The link is given by writing $\chi(0)$ in terms of momentum dependent operators:

$$
\chi(0) = \int \frac{d^3k}{k_0}(\tilde{a}(k) + \tilde{a}(k)^\dagger), \quad (17)
$$

where $\tilde{a}(k)^\dagger$ is the adjoint of $\tilde{a}(k)$ and guarantees that $\chi(0)$ is hermitian. Using the results of reference [1], Eq.33 and keeping only the annihilation operator parts, Eq.14 becomes

$$
\begin{aligned}
-B_{\mu\alpha}(k)g_{\alpha,\alpha}(a'(k,\alpha) - a(k,\alpha)) &= i[P_\mu, \tilde{a}(k)]\\
&= -k_\mu f(k)I.\\
[P_\mu, \tilde{a}(k)] &= ik_\mu f(k)I && (18)
\end{aligned}
$$

Since the four-momentum operator P_μ contains all the dynamics, Eq.18 implies that the gauge operator $\tilde{a}(k)$ also is dynamical, while the c-number gauge transformation $f(k)$ given in Eq.33 of reference [1] remains kinematical.

To conclude this section Maxwell's equations are shown to be expectation values of the operator relations. Thus,

$$
\begin{aligned}
< F_{\mu\nu}(x) > &= (\Psi_0, e^{iP\cdot x}F_{\mu\nu}(0)e^{-iP\cdot x}\Psi_0)\\
&= (\Psi_x, F_{\mu\nu}(0)\Psi_x), && (19)\\
\frac{\partial}{\partial x_\nu} < F_{\mu\nu}(x) > &= (\Psi_x, J_\mu(0)\Psi_x)\\
&= < J_\mu(x) >; && (20)\\
< F_{\mu\nu}(x) > &= \frac{\partial}{\partial x^\nu} < A_\mu(x) > - \frac{\partial}{\partial x^\mu} < A_\nu(x) >, && (21)
\end{aligned}
$$

where use has been made of the relativistic Schrödinger equation, Eq.3 and Ψ_0 is a wavefunction at the space-time point zero. Now in general $\frac{\partial}{\partial x_\mu} < A_\mu(x) > \neq 0$. But, just as in classical electrodynamics, where one may make a gauge transformation so that the four divergence of the vector potential is zero, so too the operator $\chi(0)$ can be chosen so that the delambertian of expectation values of $\chi(x)$ produces a four-divergence of expectation values of the four-vector potential being zero. The point is that for any four-momentum operator satisfying the point form equations, Eqs.1,2, an operator gauge transformation, Eq.14 results in a gauge invariant field operator, $F_{\mu\nu}(0)$, as seen in Eq.16, and a gauge invariant current operator.

4 Examples of Conserved Current Operators

As shown in the previous section, for a given four-momentum operator P_μ, there is a unique conserved current operator given in Eq.10 as the commutator of $F_{\mu\nu}(0)$ with P^μ, where $F_{\mu\nu}(0)$ itself is the commutator of $A_\mu(0)$ with P_ν (see Eq.6) and $A_\mu(0)$ is related to the photon creation and annihilation operators, as given in Eq.28 of reference [1]. Thus, in order to compute a current operator, it is necessary to specify P_μ. The goal of this section is to compute some current operators, arising both in quantum field theory and relativistic quantum mechanics.

To begin, the electromagnetic field tensor can always be written as the sum of two terms, $F_{\mu\nu}(0) = F^{(\gamma)}_{\mu\nu}(0) + F^{(em)}_{\mu\nu}(0)$, where $F^{(\gamma)}_{\mu\nu}(0) := [A_\mu(0), P_\nu(\gamma)] - [A_\nu(0), P_\mu(\gamma)]$ and $F^{(em)}_{\mu\nu}(0) := [A_\mu(0), P_\nu(em)] - [A_\nu(0), P_\mu(em)]$. If $P_\mu(em) = 0$ there is a trivial current operator associated with $F^{(\gamma)}_{\mu\nu}(0)$ which involves only photon creation and annihilation operators. This current operator, denoted by $J^{(\gamma)}_\mu(0)$, involves only timelike and longitudinal photons, which, as shown in reference [1], have zero norm. Then

$$\begin{aligned} [A_\mu(0), P_\nu] &= [A_\mu(0), P_\nu(\gamma)] \\ &= i\frac{\partial A_\mu(x)}{\partial x^\nu}|_{x=0}; \end{aligned} \tag{22}$$

$$\begin{aligned} F^{(\gamma)}_{\mu\nu}(0) &= \frac{\partial A_\mu(x)}{\partial x^\nu} - \frac{\partial A_\nu(x)}{\partial x^\mu}|_{x=0} \\ &= \int \frac{d^3k}{k_0}(B_{\mu\alpha}(k)k_\nu - B_{\nu\alpha}(k)k_\mu)(a(k,\alpha) - a^\dagger(k,\alpha)) \end{aligned} \tag{23}$$

$$
\begin{aligned}
J^{(\gamma)}_\mu(0) &= \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x_\nu}A_\mu(x)|_{x=0} + \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\nu}A_\nu(x)|_{x=0} \\
&= \sum \int \frac{d^3k}{k_0} k_\mu k^\nu B_{\nu\alpha}(k) g_{\alpha,\alpha}(a(k,\alpha) + a^\dagger(k,\alpha)) \\
&= \sum \int \frac{d^3k}{k_0} k_\mu k^{st}_\alpha a^\dagger(k,\alpha), \qquad (24)
\end{aligned}
$$

where in getting to Eq.24 use was made of the operator constraint $\sum k^{st}_\alpha g_{\alpha,\alpha} a(k,\alpha) = 0$ from Eq.26 of reference [1] and $A_\mu(x)$ is the free four-vector potential of Eq.(). Both $F^{(\gamma)}_{\mu\nu}(0)$ and $J^{(\gamma)}_\mu(0)$ act only in the photon Fock space. As can be seen from Eq.24 this current operator has only longitudinal and timelike photons. For example, $||J^{(\gamma)}_\mu(0)|0> || = 0$. Even when $P_\mu(em)$ is not zero, $F^{(\gamma)}_{\mu\nu}(0)$ will depend only on photon creation and annihilation operators. The currents arising from $F^{(\gamma)}_{\mu\nu}(0)$ can then only have the form of free current operators. Hence in the following only $F^{(em)}_{\mu\nu}(0)$ will be used to generate current operators.

Consider next an example from quantum field theory in which the matter Lagrangian is the Dirac nucleon Lagrangian, which, under mimimal substitution results in a Lagrangian, $\mathcal{L} = \mathcal{L}_{matter} + \mathcal{L}_\gamma + \mathcal{L}_{em}$ [3], resulting in a four-momentum operator P_μ which is the sum of $P_\mu(matter)$, the free four-momentum operator for electrons and positrons, $P_\mu(\gamma)$, the photon four-momentum operator given in reference [1], Eq.25, and $P_\mu(em)$, the electromagnetic four-momentum operator:

$$
P_\mu(em) := e\int d^4x\delta(x\cdot x - \tau^2)x_\mu\theta(x_0)j^\nu(x)A_\nu(x); \qquad (25)
$$

the electromagnetic four-momentum operator has been obtained by integrating $\mathcal{L}_{em}$ over the forward hyperboloid, with x_μ the four-vector perpendicular to the hyperboloid. $j^\nu(x) = \bar{\Psi}(x)\gamma^\nu\Psi(x)$ is the usual local Dirac current which appears in the field equations for such a Lagrangian (see for example Schweber, reference [3], page 276).

As stated in section 2 the fields in Eqs.7,11 are in general not local. However locality still plays an important role in the point form in that it guarantees that the four-momentum operators will satisfy Eq.1 when they are made out of free (and therefore local) fields. To see this consider the commutator $[P_\mu(em), P_\nu(em)]$:

$$
[P_\mu(em), P_\nu(em)] = e^2\int d^4xd^4y\delta(x\cdot x-\tau^2)\delta(y\cdot y-\tau^2)x_\mu\theta(x_0)
$$

$$\begin{aligned} & y_\nu \theta(y_0)[j^\alpha(x)A_\alpha(x), j^\beta(y)A_\beta(y)] \\ = \ & 0. \end{aligned} \tag{26}$$

The commutator is zero because the operators in the commutator are all local and $(x-y)^2$ is space-like (since both x and y sit on the forward hyperboloid determined by τ, the difference must be space-like). Using similar reasoning the components of the total four-momentum operator will commute among themselves and transform as a Lorentz four-vector, thus satisfying Eqs.1,2.

To compute the current associated with this four-momentum operator, it is necessary to first compute $F^{(em)}_{\mu\nu}(0)$:

$$iF^{(em)}_{\mu\nu}(0) := [A_\mu(0), P_\nu(em)] - [A_\nu(0), P_\mu(em)] \tag{27}$$

$$= \int d^4x \delta(x\cdot x - \tau^2)\theta(x_0) j^\rho(x) \{x_\nu[A_\mu(0), A_\rho(x)] - x_\mu[A_\nu(0), A_\rho(x)]\}$$

$$F^{(em)}_{\mu\nu}(0) = \int d^4x \delta(x\cdot x - \tau^2)\theta(x_0)(j_\mu(x)x_\nu - j_\nu(x)x_\mu)\Delta(x). \tag{28}$$

Here use has been made of the locality of the free four-vector potential, $[A_\mu(0), A_\nu(x)] = ig_{\mu\nu}\Delta(x)$, where $\Delta(x)$ is the invariant function (see for example Schweber, reference [3],page 242).

With these expressions for the field tensor it is now possible to compute the current operator:

$$iJ_\mu(0) = [F^{(em)}_{\mu\nu}(0), P^\nu(matter) + P^\nu(\gamma)]$$

$$= [F^{(em)}_{\mu\nu}(0), P^\nu(matter)] \tag{29}$$

$$= \int d^4x \delta(x\cdot x - \tau^2)\theta(x_0)\Delta(x) [x_\nu j_\mu(x) - x_\mu j_\nu(x), P^\nu(matter)] \tag{30}$$

$$= -i\int d^4x \delta(x\cdot x - \tau^2)\theta(x_0)\Delta(x) x_\nu \frac{\partial j_\mu(x)}{\partial x_\nu}. \tag{31}$$

Note that

$$[F^{(em)}_{\mu\nu}(0), P^\nu(em)] = [\int d^4x \delta(x\cdot x - \tau^2)\theta(x_0)\Delta(x)(j_\mu(x)x_\nu - j_\nu(x)x_\mu), \int d^4y \delta(y\cdot y - \tau^2)\theta(y_0) y^\nu j^\rho(y) A_\rho(y)] \tag{32}$$

$$= 0, \tag{33}$$

since $j_\mu(x)$ is a local operator, and x and y are space-like separated.

The conserved current operator is then

$$J_\mu(0) = \int d^4x\delta(x\cdot x - \tau^2)\theta(x_0)x^\nu \frac{\partial j_\mu(x)}{\partial x^\nu}\Delta(x)\}. \qquad (34)$$

If $j_\mu(x)$ is written in terms of nucleon and antinucleon creation and annihilation operators, Eq.34 shows that $J_\mu(0)$ has the same creation and annihilation operator structure as $j_\mu(x)$; they differ only by a form-factor. As it stands $J_\mu(0)$ is conserved because $j_\mu(0)$ with respect to the free matter four-momentum operator is conserved. However, if the matter four-momentum operator consists of free plus interacting parts, and the interacting part of the Lagrangian contains no derivative coupling terms, then $J_\mu(0)$ in Eq.34 will still be conserved while $j_\mu(0)$ is not.

The main purpose of this paper is to construct conserved current operators related to a matter four-momentum operator which is constructed using the so called Bakamjian-Thomas procedure [4]. For simplicity only two spinless, equal mass particles are considered. If p_1 and p_2 are the four-momenta of particles 1 and 2, then a velocity state [5] with four-velocity v ($v\cdot v = 1$) and internal momentum $\vec{u}$ is related to single particle states by

$$|v,\vec{u}> = |p_1>|p_2>; \qquad (35)$$

$$\begin{aligned} p_1 &= B(v)u_1 \\ p_2 &= B(v)u_2, \end{aligned} \qquad (36)$$

with $u_1 = (\omega,\vec{u}), u_2 = (\omega,-\vec{u})$ and $\omega = \sqrt{m^2+\vec{u}\cdot\vec{u}}$. Under a Lorentz transformation the velocity state transforms as

$$U_\Lambda|v,\vec{u}> = |\Lambda v, R_W\vec{u}>, \qquad (37)$$

where R_W is a Wigner rotation, $R_W = B^{-1}(\Lambda v)\Lambda B(v)$, and $B(v)$ a canonical spin boost[5], a Lorentz group matrix defined by

$$B(v) = \left[\begin{array}{cc} v_0 & \vec{v}^T \\ \vec{v} & \frac{I+\vec{v}\otimes\vec{v}^T}{1+v_0} \end{array}\right] \qquad (38)$$

and satisfying $p = B(v)p^{st}, p^{st} = (m,0,0,0)$.

Then a matter four-momentum operator is given by

$$P_\mu(matter) = MV_\mu, \qquad (39)$$

where the four-velocity operator is defined by $V_\mu|v,\vec{u}>=v_\mu|v,\vec{u}>$, and the mass operator M has a kernel that depends on internal variables only and is rotationally invariant. The operator $P_\mu(matter)$ as defined in Eq.40 satisfies the point form Eqs.1,2 and acts on a two-particle Hilbert space $\mathcal{H}_2$, consisting of square integrable functions in $\vec{p}_1$ and $\vec{p}_2$. Acting on elements in $\mathcal{H}_2, U_\Lambda$ is a unitary operator.

The tensor product of $\mathcal{H}_2$ with the photon Fock space $\mathcal{F}(\gamma)$ discussed in reference [1] generates a new four-momentum operator, the sum of $P_\mu(matter)$ $+P_\mu(\gamma)$, such that again the point form Eqs.1,2 are satisfied. If now the electromagnetic four-momentum operator $P_\mu(em)$ is defined as in Eq.25, where $j_\mu(x)$ is an arbitrary one or two-body current operator acting in $\mathcal{H}_2$, then $P_\mu = P_\mu(matter)+P_\mu(\gamma)+P_\mu(em)$ will no longer satisfy the point form equations because $j_\mu(x)$ is not a local operator. To see how the electromagnetic four-momentum operator can be modified to satisfy the point form equations, the Hilbert space is restricted to a direct sum of $\mathcal{H}_2$ and the tensor product of $\mathcal{H}_2$ with the one photon Hilbert space given in reference [1], Eq.19. The goal is to write $P_\mu(em)$ on this direct sum space in Bakamjian-Thomas form. Taking matrix elements of Eq.25 the electromagnetic four-momentum operator becomes

$$\begin{aligned}
< p_1^{'}p_2^{'}|P_\mu(em)|p_1p_2k\alpha > &= \int d^4x\delta(x\cdot x-\tau^2)x_\mu\theta(x_0)e^{i(m^{'}v^{'}-mv-k)\cdot x} \\
&\quad < p_1^{'}p_2^{'}|j^\rho(0)|p_1p_2 >< 0|A_\rho(0)|k\alpha > \\
&\approx (v^{'})_0\delta^3(v^{'}-\frac{mv+k}{\sqrt{m^2+2mv\cdot k}})v_\mu f \\
&\quad < p_1^{'}p_2^{'}|j^\rho(0)|p_1p_2 >< 0|A_\rho(0)|k\alpha >, \qquad (40)
\end{aligned}$$

where $mv = p_1+p_2+k$ and $m^{'}v^{'} = p_1^{'}+p_2^{'}$. The integral over the forward hyperboloid has been approximated by a four-velocity delta function times a form factor f, with f a function of Lorentz invartiants and having the dimensions of length cubed. Because of the four-velocity delta function the four-momentum operator now satisfies the point form Eqs. 1,2. Lorentz covariance is satisfied since $j^\rho(0)$ and $A_\rho(0)$ separately transform as Lorentz four-vectors. This can be seen more clearly by examining two particle to three particle matrix elements:

$$\begin{aligned}
< p_1^{'}p_2^{'}|V_\mu M^{em}|p_1p_2k\alpha > &= v_\mu^{'} < p_1^{'}p_2^{'}|M^{em}|p_1p_2k\alpha > |_{v^{'}=v} \qquad (41)\\
< p_1^{'}p_2^{'}|M^{em}V_\mu|p_1p_2k\alpha > &= < p_1^{'}p_2^{'}|M^{em}|p_1p_2k\alpha > |_{v^{'}=v}v_\mu, \qquad (42)
\end{aligned}$$

where, from Eq.41, $< p_1^{'} p_2^{'} | M^{em} | p_1 p_2 k\alpha > = < p_1^{'} p_2^{'} | j^\rho(0) | p_1 p_2 >< 0 | A_\rho(0) | k\alpha > f(m^{'} - m)$. The right hand sides of Eqs.42,43 are equal because the four-velocity delta function in the electromagnetic mass operator forces the four-velocity $v^{'}$ of $p_1^{'} + p_2{'}$ to be equal to the four-velocity v of $p_1 + p_2 + k$; hence the left hand sides of Eqs.45,46 are equal and the four-velocity operator commutes with the electromagnetic mass operator.

Similarly the Lorentz transformation properties, Eq.2, are satisfied if the electromagnetic mass operator commutes with Lorentz transformations:

$$
\begin{aligned}
< p_1^{'} p_2^{'} | M^{em} U_\Lambda | p_1 p_2 k\alpha > &= \sum < p_1^{'} p_2^{'} | M^{em} | \Lambda p_1, \Lambda p_2, \Lambda k, \beta > \Lambda_{\beta\alpha}(e_W) \\
&= \sum < p_1^{'} p_2^{'} | j^\rho(0) | \Lambda p_1, \Lambda p_2 > v_0^{'} \delta^3(v^{'} - \Lambda v) \\
&\quad < 0 | A_\rho(0) | \Lambda k, \beta > \Lambda_{\beta\alpha}(e_W) \qquad (43) \\
&= < \Lambda^{-1} p_1^{'}, \Lambda^{-1} p_2^{'} | j^\gamma(0) | p_1 p_2 > \Lambda^{-1}_{\gamma\rho} g_{\rho\rho} \\
&\quad (\Lambda^{-1} v')_0 \delta^3(\Lambda^{-1} v^{'} - v) \\
&\quad B_{\rho\beta}(\Lambda k) \Lambda_{\beta\alpha}(e_W) \qquad (44) \\
&= < p_1^{'} p_2^{'} | U_\Lambda M^{em} | p_1 p_2 k\alpha > . \qquad (45)
\end{aligned}
$$

These results are also valid for particles with spin; in such a case, to show Lorentz covariance, there will be additional D functions whose arguments are Wigner rotations.

On the direct sum space, the full four-momentum operator can thus be written as

$$
P_\mu = \begin{bmatrix} P_\mu(matter) & P_\mu(em) \\ P_\mu^\dagger(em) & P_\mu(3) \end{bmatrix} \qquad (46)
$$

where the matrix elements of $P_\mu(em)$ are given in Eq.40 and $P_\mu(matter)$ in Eq.39. $P_\mu(3)$ is the four-momentum operator acting in the three-particle space; its actual form is not needed for calculating two-particle current matrix elements. From the four-momentum operator given in Eq.46 it is possible to compute the (strongly) conserved current operator, $J_\mu(0)$, or better its two particle matrix elements $< p_1^{'} p_2^{'} | J_\mu(0) | p_1 p_2 >$. The field tensor is:

$$
i F_{\mu\nu}^{(em)}(0) = [A_\mu(0), P_\nu(em)] - [A_\nu(0), P_\mu(em)].
$$

which has only two (or three) particle matrix elements:

$$
< p_1^{'} p_2^{'} | F_{\mu\nu}^{(em)}(0) | p_1 p_2 > = < p_1^{'} p_2^{'} | [A_\mu(0), P_\nu(em)] - [A_\nu(0), P_\mu(em)] | p_1 p_2 >
$$

$$
\begin{aligned}
&= \; < p_1^{'}p_2^{'}|j_\mu(0)|p_1p_2> (v_\nu \frac{F^{'}}{m^{'}} - v_\nu^{'}\frac{F}{m}) \\
&\quad - < p_1^{'}p_2^{'}|j_\nu(0)|p_1p_2> (v_\mu \frac{F^{'}}{m^{'}} - v_\mu^{'}\frac{F}{m}),
\end{aligned} \tag{47}
$$

where $mv = p_1+p_2, m^{'}v^{'} = p_1^{'}+p_2^{'}$ and F (respectively $F^{'}$) is a (dimensionless) function of the invariants $m, m^{'}$ and $v \cdot v^{'}$.

Two particle matrix elements of the current operator now depend on dynamics, namely the matter mass operator:

$$
i < p_1^{'}p_2^{'}|J_\mu(0)|p_1p_2> \; = \; < p_1^{'}p_2^{'}|[F_{\mu\nu}^{(em)}(0), P^\nu(matter)]|p_1p_2> \tag{48}
$$

$$
\begin{aligned}
&= \; < p_1^{'}p_2^{'}|F_{\mu\nu}^{(em)}(0)M_{matter}|p_1p_2> v^\nu \\
&\quad - < p_1^{'}p_2^{'}|M_{matter}F_{\mu\nu}^{(em)}(0)|p_1p_2> (v^{'})^\nu,
\end{aligned} \tag{49}
$$

and is by construction conserved. The matter mass operator can be written as the sum of free and interacting mass operators, $M_{matter} = M_{free} + M_{int}$ so that the current operator becomes a sum of free and interacting current operators,

$$
\begin{aligned}
i < p_1^{'}p_2^{'}|J_\mu^{free}(0)|p_1p_2> \; &= \; < p_1^{'}p_2^{'}|F_{\mu\nu}^{em}(0)|p_1p_2> (mv^\nu - m^{'}(v^{'})^\nu) \\
&= \; < p_1^{'}p_2^{'}|j_\mu(0)|p_1p_2> (v_\nu \frac{F^{'}}{m^{'}} - v_\nu^{'}\frac{F}{m}) \\
&\quad (mv^\nu - m^{'}(v^{'})^\nu) - (mv^\nu - m^{'}(v^{'})^\nu \\
&\quad < p_1^{'}p_2^{'}|j_\nu(0)|p_1p_2> (v_\mu \frac{F^{'}}{m^{'}} - v_\mu^{'}\frac{F}{m})
\end{aligned} \tag{50}
$$

$$
\begin{aligned}
&= \; < p_1^{'}p_2^{'}|j_\mu(0)|p_1p_2> (\frac{m}{m^{'}}F^{'} + \frac{m^{'}}{m}F - v \cdot v^{'}(F^{'} + F)) \\
&\quad - < p_1^{'}p_2^{'}|v \cdot j(0)|p_1p_2> (v_\mu \frac{m}{m^{'}}F^{'} - v_\mu^{'}F) \\
&\quad - < p_1^{'}p_2^{'}|v^{'} \cdot j(0)|p_1p_2> (v_\mu F^{'} - v_\mu^{'}\frac{m^{'}}{m}F),
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
i < p_1^{'}p_2^{'}|J_\mu^{int}(0)|p_1p_2> \; &= \; < p_1^{'}p_2^{'}|F_{\mu\nu}^{em}(0)M_{int}|p_1p_2> v^\nu \\
&\quad - < p_1^{'}p_2^{'}|M_{int}F_{\mu\nu}^{em}(0)|p_1p_2> (v^{'})^\nu.
\end{aligned} \tag{52}
$$

If $j_\mu(0)$ is a one-body current operator, then Eq.52 can be evaluated explicitly in terms of matrix elements of $j_\mu(0)$ and M_{int}. No assumptions about $j_\mu(0)$ have been made; it need not be conserved even in the absence of strong interactions. The extra terms in Eq.51 guarantee that $J_\mu^{free}(0)$ is conserved even when the interacting mass operator is zero. For example, if the two particle matrix elements of $j_\mu(0)$ are a two particle to two particle Feynman diagram with an external photon line, then it is not conserved with respect to the free or the free plus interacting mass operator. But the current matrix elements in Eq.51, 52 will be conserved.

5 Conclusion

In this paper a general procedure for constructing conserved current operators has been given, for both finite and infinite degree of freedom systems. The basic operators in this construction-the four-vector potential, the field tensor, and the current operators-are no longer local fields, but fields determined by operators defined at a fixed space-time point chosen to be the zero space-time point. The operators at an arbitrary space-time point are then translated from the space-time point zero by the four-momentum operator. The point form is thus crucial for this construction, as all the interactions are in the four-momentum operators, while all the Lorentz generators are kinematic. And while the four-vector potential at zero is always kinematic (see reference [1],Eq.28), the field tensor and current operators are dynamic in that they are determined by commutators with the four-momentum operator (see Eqs.6,10).

The construction given here is to be contrasted with the more usual (instant form) quantum field theory in the following way. Given free field operators a Lagrangian is made out of polynomials of free (local) fields. In quantum field theory currents arise from internal symmetries of the Lagrangian and are conserved only if the equations of motion are satisfied. If the equations of motion are satisfied the resulting currents are both local and conserved. In contrast, in the point form as discussed in this paper there are no equations of motion, only four-momentum operators that must satisfy the point form Eqs.1,2. Operator equations of motion have been replaced by state vector equations of motion, given by the relativistic Schrödinger equation, Eq.3. Locality however still plays an important role in that it can be used to help satisfy the point form equations (see for example, Eq.25). Given a four-momentum operator that satisfies the point form equations and includes electromagnetic couplings (that is, depends on photon creation and annihilation operators) a field tensor at the space-time point zero (Eq.6) and thence a current operator (Eq.10) can be constructed as commutators. Such a current operator is always conserved.

Current operators at an arbitrary space-time point, as translates of the current operator at the space-time point zero, are not local operators. Nevertheless averages of these operators satisfy the classical Maxwell equations, as shown in Eq.20. In general, averages of the four-vector potential do not satisfy a Lorentz gauge condition, but can be made to do so by a suitable gauge transformation. And, as shown in Eq.16, the field tensor and current operators are gauge invariant. Moreover, since the field quantities are not local, gauge transformations are operator rather than c-number transformations.

The procedures given in this paper for constructing conserved current operators can be used not only for infinite degree of freedom systems, but also finite degree of freedom systems. In this latter case the starting point for the matter part of the four-momentum operator is a Bakamjian-Thomas type of construction [4], in which the matter four-momentum operator is written as a product of a four-velocity operator and a mass operator. To this a photon four-momentum operator can be added while still preserving the point form equations. If a current-say a one-body current-is then coupled to the free photon field, as in Eq.25, the resulting four-momentum operator will not satisfy the point form equations. However matrix elements of the current times four-vector potential taken at the space-time point zero will satisfy the point form equations if the matrix elements are multiplied by a four-velocity delta function, so that initial and final four velocities are the same. In this way a four-momentum operator as the product of four-velocity times mass operator will satisfy the point form equations, with the mass operator a sum of matter, photon, and electromagnetic mass operators.

The key idea for constructing conserved electromagnetic current operators for any matter interaction is that it is given as the commutator of the electromagnetic field tensor with the matter four-momentum operator. If the electromagnetic field tensor is antisymmetric in its indices, and the matter four-momentum operators commute among themselves, the resulting current will be conserved. Now usually the electromagnetic four-momentum operator is formed by coupling the four-vector potential operator to a current. This current need have no conservation properties; the electromagnetic field tensor as the commutator of the four-vector potential with the electromagnetic four-momentum will be proportional to this current. Thus, starting with a nonconserved current leads to a conserved current, which then can be used to modify the electromagnetic four-momentum operator. It should be noted that these conclusions all change if the coupling of the electromagnetic field to matter includes higher powers of the photon field, as is the case with the coupling of photons to charged mesons [3]. What the currents look like when photons are coupled to a pion-nucleon vertex will be discussed in a later paper.

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